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On spiral coordinates with application to wave propagation

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We introduce a possibly new system of orthogonal curvilinear coordinates, whose coordinate curves are logarithmic spirals in the plane, supplemented by a cylindrical coordinate for three dimensions. It is shown that plane spiral coordinates form a one-parameter family, with equal scale factors along the two orthogonal coordinate curves, and constant Christoffel symbols. The equations of magnetohydrodynamics, which include those of fluid mechanics, are written in spiral coordinates and used to find a state of magnetohydrostatic equilibrium under a radial gravity field and spiral magnetic field, and to solve the equation of non-dissipative Alfvén waves in a spiral magnetic field in terms of Bessel functions. This exact solution specifies the evolution of wave perturbations (velocity and magnetic field) and energy variables (kinetic and magnetic energy densities and energy flux) with distance, for waves of arbitrary frequency. Both the frequency and the spiral angle are varied in plots of the waveforms, which show the effect on Alfvén wave propagation of three simultaneous effects: change in the mass density of the medium and in the strength and direction of the external magnetic field.

1. Introduction

We introduce a system of plane orthogonal spiral coordinates; it can be generalized to three dimensions, like cylindrical spiral coordinates, by adding a third Cartesian coordinate, orthogonal to the plane of the first two. Although this coordinate system is fairly simple, we have no knowledge of its use in the literature; for this reason, we shall discuss in some detail the properties of this new curvilinear orthogonal coordinate system. Spiral coordinates can be arrived at in at least two distinct but equivalent ways: (*a*) physically by considering plane potential spiral flow, due to the superposition of a source/sink and a vortex, for which the equipotentials and streamlines are orthogonal logarithmic spirals; (*b*) geometrically by noting that the logarithmic spiral is the curve making a constant angle ϕ with all radial lines, and thus an orthogonal set consists of the logarithmic spirals making an angle $\pi/2-\phi$ with all radial lines.

Both approaches show that spiral coordinates are curvilinear and orthogonal, and that they form a one-parameter family, i.e. there is one set of plane curvilinear orthogonal logarithmic spiral coordinates for each value of one parameter, e.g. (*a*) geometrically, the angle ϕ one family of coordinate curves makes with the radial direction, (*b*) physically this corresponds to $\tan\phi = \Gamma/Q$, the ratio of the strength of the vortex to the strength of the source/sink. Polar coordinates correspond to the case of radial coordinate curves $\phi = 0$ or no vortex $\Gamma = 0$; thus they are equivalent to a particular case of spiral coordinates, but they are redefined here so that the two scale factors are equal. Another particular case, for which the two scale factors are obviously equal, is equilateral spiral coordinates, with coordinates curves all making the same angle $\phi = \pi/4 = \pi/4 - \phi$ with the radial direction, i.e. a source of the same strength as the vortex, $Q = \pm\Gamma$. It should be emphasized that all spiral coordinates (for any value of ϕ or $\tan\phi = \Gamma/Q$) have the same scale factor along the two coordinate curves; this is the case for all conformal coordinate systems (see the Appendix).

When plane curvilinear orthogonal spiral coordinates are generalized to three dimensions by adding a third, Cartesian, coordinate orthogonal to the plane of the first two, the third scale factor is unity, and is distinct from the other two. The extension from plane spiral to cylindrical spiral coordinates is straightforward, and we shall restrict ourselves to the former, e.g. when writing the general equations of non-dissipative magnetohydrodynamics in spiral coordinates. The equations of fluid dynamics, or magnetofluid dynamics, in spiral coordinates should be appropriate to the solution of problems with spiral geometry, e.g. flow in certain types of centrifugal turbomachinery, phenomena in spiral galaxies or diffusion along spiral trajectories. We give as an example the propagation of waves along a spiral, such as would occur for Alfvén waves in a spiral magnetic field. In this case the external magnetic field decays in magnitude along a logarithmic spiral, so as to conserve magnetic flux; the velocity and magnetic field perturbations are parallel and transverse, and thus lie along the orthogonal logarithmic spiral. The equations of magnetohydrodynamics are exactly linear in this case; although the problem is one of wave propagation with varying speed and direction, it depends only on time and one spiral coordinate, and it can be solved exactly in terms of Bessel functions.

Spiral coordinates are relevant to the geometry of the solar magnetic field, which decays in strength with distance and changes in direction along Parker's spiral; although the latter is not a logarithmic spiral, the angle Parker's spiral makes with radial lines varies slowly, i.e. on a lengthscale much larger than the local wavelength. For such wavelengths, Parker's spiral can be replaced locally by a logarithmic spiral. This is simpler than trying to use curvilinear coordinates because the orthogonal curves are not Parker's spirals, and the scale factors are more complicated than for a logarithmic spiral, leading to a problem which is more cumbersome analytically. For wavelengths smaller than the lengthscale of change of spiral angle, the two approaches are physically equivalent, and thus we follow the mathematically simpler method of using spiral coordinates as an approximation. This approximation does not prevent us from studying the effects of the change in magnitude and direction of propagation speed on wave reflection, leading to: (i) a different waveform for velocity and magnetic field perturbations, implying that equipartition of kinetic and magnetic energies is violated; (ii) non-conservation of the energy flux, and changes in the spectrum with distance. We do not attempt to model other features of the solar wind, e.g. the presence of a background flow.

2. Families of curvilinear orthogonal logarithmic spiral coordinates

One approach to plane spiral coordinates (we use this as an abbreviation of the exact designation in the title of this section) is to consider the plane spiral potential flow, for which the complex potential is specified by the superposition of a source $Q > 0$ or sink $Q < 0$ of strength Q and a vortex of circulation Γ (clockwise for $\Gamma > 0$ and counterclockwise for $\Gamma < 0$):

$$f(z) = \{(Q - i\Gamma)/2\pi\} \log z. \tag{1}$$

The potential Φ and stream function Ψ in polar coordinates (r, θ),

$$f = \Phi + i\Psi, \quad z = r\,e^{i\theta}, \tag{2a,b}$$

show that both the equipotentials $\Phi = \text{const.}$ and streamlines $\Psi = \text{const.}$ are logarithmic spirals:

$$2\pi\Phi = Q\log r + \Gamma\theta, \quad 2\pi\Psi = Q\theta - \Gamma\log r, \tag{3a, b}$$

Since they are orthogonal, they form a curvilinear orthogonal coordinate system

$$\alpha = \theta - k\log r, \quad \beta = \log r + k\theta, \tag{4a, b}$$

where (i) α, β are designated spiral coordinates because the curves $\alpha = \text{const.}$, $\beta = \text{const.}$ are orthogonal logarithmic spirals

$$\{\alpha, \beta\} \equiv (2\pi/Q)\{\Psi, \Phi\}, \tag{5a}$$

and (ii) there is a parameter

$$k \equiv \Gamma/Q, \tag{5b}$$

which can be chosen freely, so that we have a family of spiral coordinates.

The preceding results can be given a purely geometrical interpretation: (i) the coordinate curves $\alpha = \text{const.}$ in (4a), make a constant angle ϕ with all radial lines

$$(r\,\mathrm{d}\theta/\mathrm{d}r)_\alpha = k = \tan\phi, \tag{6a}$$

and hence are logarithmic spirals; (ii) the coordinate curves $\beta = \text{const.}$ in (4b) are also logarithmic spirals

$$(r\,\mathrm{d}\theta/\mathrm{d}r)_\beta = -1/k = -\cot\phi = \tan(\pi/2 - \phi), \tag{6b}$$

which make an angle $\pi/2 - \phi$ with all radial lines and are orthogonal to the coordinate curves $\alpha = \text{const.}$; (iii) we have a distinct member of the family of spiral coordinates for each angle ϕ, which is equivalent to

$$\Gamma/Q = \tan\phi, \tag{7}$$

the ratio of vortex and source/sink strengths. In the particular case of a source/sink alone (i.e. no vortex $\Gamma = 0$), we obtain a system equivalent to polar coordinates:

$$k = 0: \quad \alpha = \theta, \quad \beta = \log r, \tag{8a, b}$$

with the radial coordinates re-defined, so that the arclength

$$(\mathrm{d}l)^2 = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2, \tag{9}$$

has the same scale factor

$$(\mathrm{d}l)^2 = r^2\{(\mathrm{d}\log r)^2 + (\mathrm{d}\theta)^2\} = r^2\{(\mathrm{d}\alpha)^2 + (\mathrm{d}\beta)^2\} \tag{10}$$

in the α- and β-directions.

Another particular case is equilateral spiral coordinates, when both sets of coordinates curves make the same angle with all radial lines:

$$\phi = \pi/4 = \pi/2 - \phi, \quad k = 1, \quad |Q| = |\Gamma|, \tag{11a–c}$$

corresponding to spiral flow with a source/sink of the same strength, in modulus, as the vortex. In this case,

$$\alpha = \theta - \log r, \quad \beta = \theta + \log r, \tag{12a, b}$$

we should expect the scale factor to be same in both directions:

$$(\mathrm{d}l)^2 = r^2\{(\mathrm{d}\log r)^2 + (\mathrm{d}\theta)^2\} = \tfrac{1}{2}r^2\{(\mathrm{d}\alpha)^2 + (\mathrm{d}\beta)^2\}, \tag{13}$$

although $s = r$ in (10) and $s = \frac{1}{2}r^2$ in (13). We note that the scale factor is the same in both coordinate directions for all spiral coordinates, as must be the case for any orthogonal coordinate system arising out of a conformal mapping (see the Appendix). We calculate the value of the scale factor as follows: (i) first we invert the transformation ($4a, b$) between polar and spiral coordinates:

$$(1+k^2)(\theta, \log r) = \{\alpha + k\beta, \beta - k\alpha\}; \tag{14}$$

(ii) substituting in the arclength in polar coordinates (9), we obtain the arclength in spiral coordinates:

$$(\mathrm{d}l)^2 = \{r^2/(1+k^2)\}[(\mathrm{d}\alpha)^2 + (\mathrm{d}\beta)^2]. \tag{15}$$

Note that (15) reduces to (10) for $k = 0$, and to (13) for $k = 1$.

The general Riemannian two-dimensional curvilinear arclength is

$$(\mathrm{d}l)^2 = g_{\alpha\alpha}(\mathrm{d}\alpha)^2 + 2g_{\alpha\beta}\,\mathrm{d}\alpha\,\mathrm{d}\beta + g_{\beta\beta}(\mathrm{d}\beta)^2, \tag{16}$$

where g_{ij} designates the metric tensor; comparing (16) with (15) it follows that the non-diagonal elements vanish, $g_{\alpha\beta} = 0$, which provides a third proof that spiral coordinates are orthogonal. The diagonal elements specify the scale factors

$$g_{\alpha\alpha} = g_{\beta\beta} = r^2/(1+k^2) \equiv s^2, \tag{17}$$

and they coincide, so that we have the same scale factor

$$s \equiv r/(1+k^2)^{1/2} = (1+k^2)^{-1/2} \exp\{(\beta - k\alpha)/(1+k^2)\} \tag{18}$$

along both coordinate curves. One possible extension from two to three dimensions, preserving orthogonality, is to add a third, Cartesian coordinate z:

$$(\mathrm{d}l)^2 = s^2\{(\mathrm{d}\alpha)^2 + (\mathrm{d}\beta)^2\} + (\mathrm{d}z)^2, \tag{19}$$

whose scale factor is distinct (namely unity). We designate (α, β, z) cylindrical spiral coordinates, because: (i) the coordinate surfaces $\alpha, \beta = \text{const.}$ are cylinders with orthogonal spirals as directrices, and generators normal to the plane of the spirals: (ii) $z = \text{const.}$ is a plane; (iii) the coordinate curves ($\alpha = \text{const.}$, $z = \text{const.}$) and ($\beta = \text{const.}$, $z = \text{const.}$) are orthogonal spirals in the plane $z = \text{const.}$; (iv) the coordinate curve ($\alpha = \text{const.}$, $\beta = \text{const.}$) is a straight line, i.e. the common generator of the orthogonal spiral cylinders.

3. Equations of magnetohydrodynamics in spiral coordinates

Since cylindrical spiral coordinates are a straightforward extension of plane spiral coordinates, we concentrate henceforth on the latter, using the following short-hand notation for derivatives:

$$\{\dot{F}, \partial_\alpha F, \partial_\beta F\} \equiv \{\partial F/\partial t, \partial F/\partial \alpha, \partial F/\partial \beta\}. \tag{20}$$

The fact that spiral coordinates (like all conformal coordinates; see Appendix) have the same scale factor along both coordinate curves simplifies the usual invariant differential operators, e.g. the gradient

$$\boldsymbol{\nabla}\Phi = s^{-1}(\boldsymbol{e}_\alpha \partial_\alpha \Phi + \boldsymbol{e}_\beta \partial_\beta \Phi) \tag{21}$$

and Laplacian

$$\nabla^2 \Phi = s^{-2}\{\partial_\alpha \partial_\alpha \Phi + \partial_\beta \partial_\beta \Phi\} \tag{22}$$

of a scalar $\Phi \equiv \Phi(\alpha, \beta)$. We also have the divergence

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Psi} = s^{-2}\{\partial_\alpha(s\Psi_\alpha) + \partial_\beta(s\Psi_\beta)\}, \tag{23}$$

the curl

$$\boldsymbol{\nabla} \wedge \boldsymbol{\Psi} = s^{-2}\{\partial_\alpha(s\Psi_\beta) - \partial_\beta(s\Psi_\alpha)\}\,\bar{e}_z, \tag{24}$$

and Laplacian

$$\nabla^2 \boldsymbol{\Psi} = \boldsymbol{e}_\alpha s^{-1} \partial_\alpha \{s^{-2} [\partial_\alpha (s\Psi_\alpha) + \partial_\beta (s\Psi_\alpha)]\} - \boldsymbol{e}_\alpha s^{-1} \partial_\beta \{s^{-2} [\partial_\alpha (s\Psi_\beta) - \partial_\beta (s\Psi_\alpha)]\} + \boldsymbol{e}_\beta s^{-1} \partial_\beta \{s^{-2} [\partial_\alpha (s\Psi_\alpha) + \partial_\beta (s\Psi_\beta)]\} + \boldsymbol{e}_\beta s^{-1} \partial_\alpha \{s^{-2} [\partial_\alpha (s\Psi_\beta) - \partial_\beta (s\Psi_\alpha)]\}, \quad (25)$$

for a vector $\boldsymbol{\Psi} = \Psi_\alpha \boldsymbol{e}_\alpha + \Psi_\beta \boldsymbol{e}_\beta$. The notation should be clear, e.g. $\partial_\alpha \Psi_\alpha \equiv \partial \Psi_\alpha / \partial \alpha$.

We take as the first equation of fluid mechanics (Landau & Lifshitz 1953; Batchelor 1967) that of continuity:

$$\partial \rho / \partial t + \nabla \cdot (\rho \boldsymbol{v}) = 0, \quad (26)$$

where ρ is the mass density, and $\boldsymbol{v}$ is the velocity; in spiral coordinates it reads

$$s^2 \dot{\rho} + \partial_\alpha (s v_\alpha \rho) + \partial_\beta (s v_\beta \rho) = 0. \quad (27)$$

Since we will be considering non-dissipative waves, in the energy equation, we omit thermal conduction, viscosities and electrical diffusivity. Thus the equation of energy, in terms of temperature T:

$$0 = \mathrm{d}T/\mathrm{d}t \equiv \partial T / \partial t + (\boldsymbol{v} \cdot \nabla)\, T, \quad (28)$$

reads

$$s\dot{T} + v_\alpha \partial_\alpha T + v_\beta \partial_\beta T = 0 \quad (29)$$

in spiral coordinates.

Since we have omitted the Joule effect in the energy equation, we also do so in the induction equation, in non-dissipative form (Landau & Lifshitz 1956, 1966):

$$\partial \boldsymbol{H} / \partial t + \nabla \wedge (\boldsymbol{H} \wedge \boldsymbol{v}) = 0, \quad (30)$$

whose two components read

$$s\dot{H}_\alpha + \partial_\beta (H_\alpha v_\beta - H_\beta v_\alpha) = 0, \quad s\dot{H}_\beta - \partial_\alpha (H_\alpha v_\beta - H_\beta v_\alpha) = 0, \quad (31\,a, b)$$

in spiral coordinates.

The set of fundamental equations of non-dissipative magnetohydrodynamics includes, besides the preceding ones and the equation of state $p = p(\rho, T)$, the inviscid momentum equation:

$$\rho \boldsymbol{a} + \nabla p + (\mu / 4\pi)\, \boldsymbol{H} \wedge (\nabla \wedge \boldsymbol{H}) - \rho \boldsymbol{g} = 0, \quad (32a)$$

including magnetic and gravity forces. Here the acceleration is defined in curvilinear coordinates using covariant differentiation (Schouten 1956), namely for contravariant components of the acceleration

$$a^i \equiv \mathrm{d}v^i / \mathrm{d}t = (\partial v^i / \partial t) + (\nabla_j v^i)(\mathrm{d}x^j / \mathrm{d}t) = \dot{v}^i + v^j \partial_j v^i + \Gamma^i_{jk} v^j v^k, \quad (32b)$$

where ∇_j is the covariant derivative and $\partial_j \equiv \partial / \partial x^j$ the partial derivative; for physical components of the acceleration

$$a_i \equiv s a^i = \dot{v}_i + v_j \partial_j (v_i / s) + s^{-1} \Gamma^i_{jk} v_j v_k, \quad (32c)$$

where the scale factor s is the same for both directions in the case of spiral coordinates (or any other system of conformal coordinates: see the Appendix). The acceleration consists of local, convective and centripetal terms, the latter involving the Christoffel symbols in spiral coordinates, which are given by

$$\Gamma^\alpha_{\alpha\alpha} = \Gamma^\beta_{\alpha\beta} = -\Gamma^\alpha_{\beta\beta} = s^{-1} \partial_\alpha s = -k/(1+k^2), \quad (33a)$$

$$\Gamma^\beta_{\beta\beta} = \Gamma^\alpha_{\alpha\beta} = -\Gamma^\beta_{\alpha\alpha} = s^{-1} \partial_\beta s = 1/(1+k^2), \quad (33b)$$

Note that for orthogonal curvilinear coordinates with equal scale factors in all directions, the Christoffel symbols are logarithmic derivatives of the scale factor; since the scale factor is an exponential (18) of the spiral coordinates, all Christoffel symbols are constant. The property of spiral coordinates of having constant Christoffel symbols, would simplify the equations of the gravitational field (Einstein 1916; Eddington 1924; Tolman 1934; Synge 1966; Landau & Lifshitz 1966). In the present case, it simplifies the centripetal acceleration in the two physical components of the equation of momentum in spiral coordinates:

$$\rho\{s\dot{v}_\alpha + v_\alpha \partial_\alpha v_\alpha + v_\beta \partial_\beta v_\alpha\} + \rho\{v_\alpha v_\beta + k(v_\beta)^2\}/(1+k^2) + \partial_\alpha p + (\mu/4\pi) s^{-1} H_\beta\{\partial_\alpha(sH_\beta) - \partial_\beta(sH_\alpha)\} - \rho g_\alpha = 0 \quad (34a)$$

and

$$\rho\{s\dot{v}_\beta + v_\alpha \partial_\alpha v_\beta + v_\beta \partial_\beta v_\beta\} - \rho\{(v_\alpha)^2 + k v_\alpha v_\beta\}/(1+k^2) + \partial_\beta p - (\mu/4\pi) s^{-1} H_\alpha\{\partial_\alpha(sH_\beta) - \partial_\beta(sH_\alpha)\} - \rho g_\beta = 0. \quad (34b)$$

4. Magnetohydrostatic equilibrium in a spiral magnetic field

We consider next transverse waves along a spiral magnetic field (figure 1), for which the external magnetic field is tangent to a spiral, and the velocity and magnetic field perturbations are tangent to the orthogonal spiral:

$$\boldsymbol{H} = B(\beta)\,\boldsymbol{e}_\beta + h(\beta, t)\,\boldsymbol{e}_\alpha, \quad \boldsymbol{v} = v(\beta, t)\,\boldsymbol{e}_\alpha, \quad (35a, b)$$

Note that the divergence (23) of the velocity (35*b*) is given by

$$\nabla \cdot \boldsymbol{v} = s^{-2}\partial_\alpha[sv(\beta, t)] = vs^{-2}\partial_\alpha s = -kv/[s(1+k^2)], \quad (35c)$$

so that the motion is incompressible only in the case $k = 0$ of cylindrical waves, and not in the case $k \neq 0$ of spiral waves: we still call these waves Alfvén waves, since they are transversal in the cylindrical case, but emphasize that otherwise these are compressive Alfvén modes. In the case (35*a*, *b*) the equations of induction and momentum are linear (the prime denotes derivative with regard to β, e.g. $v' \equiv \partial v/\partial\beta$):

$$s\dot{h} = (Bv)', \quad s\dot{v} = (\mu B/4\pi\rho)(sh)', \quad (36a, b)$$

using the two assumptions:

$$s\dot{v}(1+k^2) \gg v^2, \quad p + \mu h^2/2\pi = \text{const.}, \quad (37a, b)$$

namely that: (i) the centripetal acceleration is negligible relative to the local acceleration (37*a*); (ii) the total pressure, gas plus magnetic, is constant (37*b*). We may use (36*a*, *b*), to obtain the Alfvén wave equations

$$s^2\ddot{v} = (\mu B/4\pi\rho)(Bv)'', \quad s\ddot{h} = \{(\mu B^2/4\pi\rho)s^{-2}(sh)'\}', \quad (38a, b)$$

for the velocity and magnetic field perturbations respectively.

Since the properties of the medium do not depend on time, it is convenient to use a Fourier decomposition

$$v, h(\beta, t) = \int_{-\infty}^{+\infty} F, G(\beta, \omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \quad (39a, b)$$

and determine the spatial dependence from the solution of the ordinary differential equation for the velocity spectrum:

$$F'' + 2\frac{B'}{B}F' + \left(\frac{\omega^2 s^2}{A^2} + \frac{B''}{B}\right)F = 0, \quad (40a)$$

where we have introduced the Alfvén speed

$$A^2 \equiv \mu B^2/4\pi\rho. \quad (40b)$$

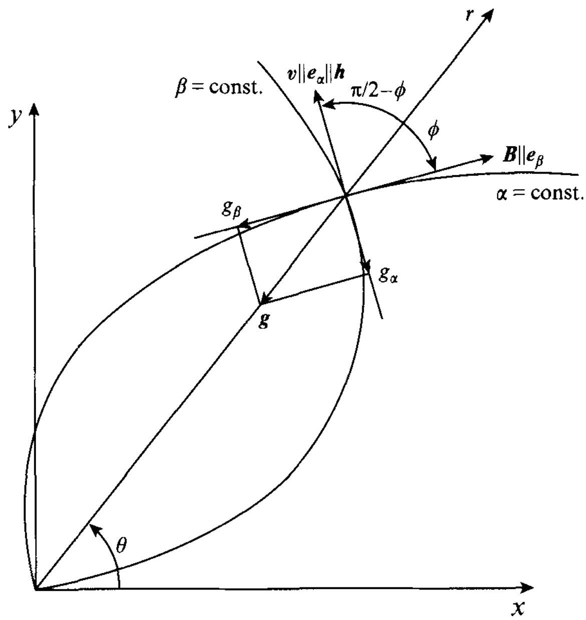


FIGURE 1. Spiral coordinates in the plane with Alfvén waves propagating along the external magnetic field, aligned with one spiral, with transverse velocity and magnetic field perturbations, aligned with the orthogonal spiral, in the presence of a radial gravity field.

We need not solve the corresponding wave equation for the magnetic field perturbation spectrum:

$$G''+2\frac{A'}{A}G'+\left[\frac{\omega^2s^2}{A^2}+\frac{s}{A^2}\left(A^2\frac{s'}{s^2}\right)'\right]G=0, \tag{41a}$$

because the latter is related to the velocity perturbation spectrum by

$$G(\beta;\omega)=(\mathrm{i}/\omega)\,(s(\beta))^{-1}\,\mathrm{d}\{B(\beta)\,F(\beta;\omega)\}/\mathrm{d}\beta, \tag{41b}$$

which follows from the polarization relation (36*a*).

Since the coefficients of the wave equation depend on the background state, we must specify the latter; namely the variation of the external magnetic field and mass density with distance which satisfies magnetohydrostatic equilibrium. The total magnetic field is divergence free (see (23)), as required by Maxwell's equation

$$0=\nabla\cdot\boldsymbol{H}=\nabla\cdot(B\boldsymbol{e}_\beta)=s^{-2}(sB)' \tag{42a}$$

if the background magnetic field decays along the spiral on the inverse of the scale factor (28):

$$B(\beta)/b=s_0/s(\beta)=\exp\{-(\beta-\beta_0)/(1+k^2)\}, \tag{42b}$$

where b denotes the magnetic field at coordinate $\beta=\beta_0$, along a spiral of constant α:

$$\alpha=\text{const.}:\quad b\equiv B(\beta_0),\quad s_0\equiv s(\beta_0). \tag{43a, b}$$

The mass density (32*a*) is determined by the condition of magnetohydrostatic equilibrium:

$$\nabla p+(\mu/4\pi)\,\boldsymbol{B}\wedge(\nabla\wedge\boldsymbol{B})=\rho\boldsymbol{g}, \tag{44a}$$

together with the equation of state

$$p=\rho RT, \tag{44b}$$

for a perfect gas of constant R. The background magnetic field is force free:

$$\boldsymbol{B} = B(\beta)\,\boldsymbol{e}_\beta: \quad \partial_\alpha p = \rho s g_\alpha, \quad \partial_\beta p = \rho s g_\beta, \tag{45a, b}$$

so we have in fact hydrostatic equilibrium. One way to integrate (45 *a*, *b*) is to note that the differential of pressure,

$$\mathrm{d}p = (\partial_\alpha p)\,\mathrm{d}\alpha + (\partial_\beta p)\,\mathrm{d}\beta = (ps/RT)\,(g_\alpha\,\mathrm{d}\alpha + g_\beta\,\mathrm{d}\beta), \tag{46}$$

must be an exact differential. To prove this we note that the gravity field is radial, and hence (figure 1) has spiral components

$$k = \tan\phi: \quad \{g_\alpha, g_\beta\} = g\{\sin\phi, -\cos\phi\} = (g/(1+k^2)^{1/2})\{k, -1\}. \tag{47a}$$

Using the scale factor (18) for spiral coordinates together with (47 *a*), in (46), leads to the simple result

$$-(RT/pg)\,\mathrm{d}p = s(k\,\mathrm{d}\alpha - \mathrm{d}\beta)/(1+k^2)^{1/2} = \mathrm{d}s(1+k^2)^{1/2} = \mathrm{d}r. \tag{47b}$$

The gravity field is also divergence free because we assume it is created by a central mass:

$$0 = \nabla\cdot\boldsymbol{g} = \nabla\cdot(g\boldsymbol{e}_r) = r^{-1}\,\partial(rg)/\partial r, \tag{48a}$$

and thus decays like the inverse of distance:

$$g = g_0(r_0/r) = g_0(s_0/s), \tag{48b}$$

or like the inverse of the scale factor (18). Substituting (48 *b*) in (47 *b*) we obtain

$$p^{-1}\,\mathrm{d}p = -(g_0 r_0/RT)\,r^{-1}\,\mathrm{d}r, \tag{49a}$$

showing that in isothermal conditions

$$T = \text{const.}: \quad p/p_0 = (r/r_0)^{-g_0 r_0/RT} = (s/s_0)^{-g_0 r_0/RT} = \rho/\rho_0, \tag{49b}$$

the gas pressure or mass density are polytrophic functions of radial distance or scale factor.

Another way to obtain this result is to note that the gravity field (48 *a*, *b*) is the gradient of a potential:

$$g = -\mathrm{d}\phi/\mathrm{d}r, \quad \phi(r) \equiv -g_0 r_0 \log r. \tag{50a, b}$$

For a force-free magnetic field and isothermal perfect gas from (44 *a*, *b*),

$$0 = p^{-1}\nabla p - g/RT = \nabla\{\log p - \phi(r)/RT\}, \tag{51}$$

it follows that

$$RT\log(p/p_0) = \phi(r) - \phi(r_0) = -g_0 r_0 \log(r/r_0), \tag{52}$$

implying (49 *b*). It follows from (42 *b*) that the Alfvén speed (40 *b*) varies like a power of distance:

$$A = a(s/s_0)^{-\nu+1}, \quad a \equiv A(s_0), \tag{53a, b}$$

with the exponent

$$\nu = 2 - g_0 r_0/2RT = 2 - r_0/2L, \tag{54a}$$

which involves the scale height

$$L \equiv RT/g_0. \tag{54b}$$

5. Wave propagation along a logarithmic spiral

A general method of solution of the linearized MHD equations (§3) for spiral coordinates in two dimensions would be to expand in the neighbourhood of a regular singularity in powers of e^α, e^β, e.g. $\mathrm{e}^{n\alpha}$, $\mathrm{e}^{m\beta} \sim s^n$, s^m, as a double series with matrix coefficients. In the one-dimensional case considered before (§4), the method simplifies to a single series of powers of s, which can be expressed in terms of Bessel functions, as we proceed to show. Substituting the external magnetic field (42*b*) and Alfvén speed (53*a*) into the wave equation (40*a*) for the velocity perturbation spectrum, we obtain

$$F'' - 2(s'/s)F' + [(s'/s)^2 + (s_0\,\omega/a)^2(s/s_0)^{2\nu}]F = 0, \tag{55}$$

where the logarithmic derivative of the scale factor (18) is a constant related to the spiral curvature:

$$s'/s = (\log s)' = 1/(1+k^2), \tag{56a}$$

and we also introduce a dimensionless frequency

$$\Omega \equiv (1+k^2)(s_0\,\omega/a\nu), \tag{56b}$$

into

$$(1+k^2)F'' - 2(1+k^2)F' + [1 + \Omega^2\nu^2(s/s_0)^{2\nu}]F = 0. \tag{57}$$

This second-order differential equation has power coefficients, which can be reduced to a quadratic via the change of independent variable:

$$\zeta = (s/s_0)^\nu = \exp\{\nu(\beta-\beta_0)/(1+k^2)\}, \tag{58a}$$

and we also make a change of dependent variable with a free parameter μ, which we may choose at will:

$$F(\beta;\omega) = \zeta^\mu I(\zeta), \tag{58b}$$

Performing the change of independent variable (58*a*), we confirm that we obtain a second-order differential equation with coefficients of second degree:

$$\{\zeta^2\,\mathrm{d}^2/\mathrm{d}\zeta^2 + (1-2/\nu)\,\zeta\mathrm{d}/\mathrm{d}\zeta + 1/\nu^2 + \Omega^2\zeta^2\}\,\zeta^\mu I(\zeta) = 0, \tag{59a}$$

and the change of dependent variable (58*b*), namely

$$\zeta^2 I'' + (2\mu + 1 - 2/\nu)\,\zeta I' + \{\Omega^2\zeta^2 + 1/\nu^2 + \mu(\mu - 2/\nu)\}\,I = 0, \tag{59b}$$

allows us to specify one coefficient, e.g. of I', by choice of μ.

For example, (59*b*) reduces to a Bessel equation of order q:

$$\zeta^2 I'' + \zeta I' + (\Omega^2\zeta^2 - q^2)\,I = 0, \tag{60}$$

if we choose for μ the value

$$\mu = 1/\nu, \tag{61a}$$

in which case it turns out that the order q of the Bessel equation

$$q = [\mu(2/\nu - \mu) - 1/\nu^2]^{1/2} = 0 \tag{61b}$$

is zero. The solution of the Bessel equation is a linear combination, either of Bessel and Neumann functions of order zero representing standing waves, or of Hankel functions of first/second kinds representing outward/inward propagating waves:

$$Z_0(\zeta) = C_1 J_0(\zeta) + C_2 Y_0(\zeta) = C_- H_0^{(1)}(\zeta) + C_+ H_0^{(2)}(\zeta), \tag{62a, b}$$

respectively, where C_1, C_2 and $C_\pm$ are arbitrary constants of integration, to be determined from boundary, asymptotic or radiation conditions. Thus the solution of (60) is

$$I(\zeta) = Z_0(\Omega\zeta), \tag{63a}$$

and the velocity perturbation spectrum

$$F(\beta;\zeta) = \zeta^{1/\nu} Z_0(\Omega\zeta) \tag{63b}$$

is given by

$$F(\beta;\omega) = (s/s_0)\, Z_0(\Omega(s/s_0)^{\nu}). \tag{64}$$

The magnetic field perturbation spectrum follows from (41 *b*), where we use (42*b*) and (56*a*):

$$G(\beta;\omega) = \mathrm{i}(B/\omega)\,\mathrm{d}(F/s)/\mathrm{d}\beta = -\mathrm{i}(B/\omega)\,(1+k^2)^{-1} s\, \mathrm{d}(F/s)/\mathrm{d}s, \tag{65}$$

and the properties of the differentiation of Bessel functions, to obtain

$$G(\beta;\omega) = \mathrm{i}(b/a)\,(s/s_0)^{\nu-1}\, Z_1(\Omega(s/s_0)^{\nu}). \tag{66}$$

This result could have been obtained by reducing (41 *a*) to a Bessel equation of order 1, as done above in (55)–(63) for (40*a*), with $\mu = 1-1/\nu$ in this case; the use of the polarization relation is simpler, and has the further advantage of specifying the constant factor ib/a in (66).

The preceding solutions of the wave equation are exact, in the sense that they are valid for all frequencies and distances, and they simplify when the Bessel function has either a small or large argument. In the case of a large variable, the asymptotic forms of propagating waves:

$$H_n^{(1,2)}(\zeta) \sim (1/\pi\zeta)^{1/2} \exp\{\pm \mathrm{i}(\zeta - n\pi/2 - \pi/4)\}, \tag{67}$$

lead to velocity and magnetic field perturbation spectra:

$$F \sim (2/\pi\Omega)^{1/2} (s/s_0)^{1-\nu/2} \exp\{\pm \mathrm{i}[\Omega(s/s_0)^{\nu} - \pi/4]\}, \tag{68a}$$

$$G \sim \mathrm{i}(b/a)\,(2/\pi\Omega)^{1/2} (s/s_0)^{(\nu/2)-1} \exp\{\pm \mathrm{i}[\Omega(s/s_0)^{\nu} - 3\pi/4]\}, \tag{68b}$$

which are related by

$$s \gg s_* \equiv s_0\,\Omega^{-1/\nu}: \quad |G| \sim \mathrm{i}(b/a)\,|F|\,(s/s_0)^{\nu-2}, \tag{69}$$

at large distance $s \gg s_*$. For a small variable, we use the initial forms of bounded standing modes:

$$J_n(\zeta) \sim (\zeta/2)^n, \tag{70}$$

which lead to velocity and magnetic field spectra:

$$F \sim s/s_0, \quad G \sim \mathrm{i}(b/a)\,(\Omega/2)\,(s/s_0)^{2\nu-1}, \tag{71a, b}$$

related by

$$s \ll s_*: \quad G \sim \mathrm{i}(b/a)\,(\Omega/2)\,(s/s_0)^{2\nu-2} F, \tag{72}$$

at small distances. The general wave field (62 *a*, *b*) is a linear combination of either of two representations, and we have used propagating waves at large distance (68 *a*, *b*) and (69), and standing waves at small distance (71 *a*, *b*) and (72), for purely analytical convenience, in displaying the results.

An Alfvén wave propagates kinetic (68 *a*) and magnetic (68 *b*) energies:

$$E_{\nu} = \tfrac{1}{2}\rho\,|F|^2, \quad E_h = (\mu/8\pi)\,|G|^2, \tag{73a, b}$$

and in addition to the total energy density

$$E = E_{\nu} + E_h, \tag{74a}$$

we mention the energy flux,

$$D = (\mu/4\pi)\,Bs\,|F|\,|G|, \tag{74b}$$

which together with the former satisfy the energy equation:

$$\partial E/\partial t + s^{-2}\,\partial D/\partial \beta = 0. \tag{75}$$

Equation (75) can be obtained from the induction (36*a*) and momentum (36*b*) equations, via the usual manipulations. We note the scaling of the mass density (49*b*) and (54*a*):

$$\rho = \rho_0 (s/s_0)^{2\nu-4}, \tag{76}$$

which is valid for all distances. For small distances and bounded J_0 standing modes, the kinetic and magnetic energies scale differently:

$$E_v \sim (s/s_0)^{2\nu-2}, \quad E_h \sim (s/s_0)^{4\nu-2}, \tag{77a, b}$$

so there is no equipartition, and the energy flux

$$D \sim (s/s_0)^{2\nu} \mp EA \tag{77c}$$

does not scale as the total energy multiplied by the Alfvén speed; the reason is that in this case $\Omega \ll 1$, and the ray approximation does not hold, i.e. the local wavelength $\lambda = 2\pi a/\omega$ is not small compared to the scale s_0 of variation of background properties. This condition,

$$1 \ll (s_0/\lambda)^2 = (s_0\,\omega/2\pi a)^2 = (1+k^2)^{-2}(\nu\Omega/2\pi)^2, \tag{78}$$

is met at large distance $s \gg s_*$, and for propagating waves there is equipartition of kinetic and magnetic energies:

$$E_v \sim (s/s_0)^{\nu-1} \sim E_h, \tag{79a}$$

and the energy flux is constant,

$$1 \sim D \sim EA, \tag{79b}$$

and scales like the energy density (79*a*) multiplied by the Alfvén speed (53*a*).

6. Scaling of wave variables and energy density and flux

The scaling of wave variables, namely velocity and magnetic field, and energies, namely kinetic and magnetic, with distance has shown an evolution from a non-ray condition $s \ll s_*$ when the non-uniformity of the medium reflects waves, to a ray condition $s \gg s_*$, when no further reflection occurs. These reflections of waves by the medium depend on frequency, and thus change the spectrum, as we will confirm by reconsidering the scaling of wave variables and energies, this time as function of frequency. We consider the velocity perturbation spectrum for a wave of frequency ω at distance s, namely for a bounded J_0 standing wave:

$$F(s;\omega) = C(\omega)\,(s/s_0)^{1/2}\,J_0(\Omega(s/s_0)^{\nu}), \tag{80a}$$

namely we select one term of (62*a*) by this boundary condition at $\zeta = 0$, or $s = 0$, or $r = 0$. The remaining constant of integration is determined from the velocity perturbation spectrum at initial position $s = s_0$:

$$F(s_0;\omega) = C(\omega)\,J_0(\Omega); \tag{80b}$$

thus the velocity (80*a*, *b*) and magnetic field perturbation (66) spectra are given by

$$F(s;\omega) = F(s_0;\omega)\,(s/s_0)\,\{J_0(\Omega(s/s_0)^{\nu})/J_0(\Omega)\}, \tag{81a}$$

$$G(s;\omega) = \mathrm{i}(b/a)\,F(s_0;\omega)\,(s/s_0)^{\nu-1}\,\{J_1(\Omega(s/s_0)^{\nu})/J_1(\Omega)\}, \tag{81b}$$

for all distances s and frequencies ω.

At large distance $s \gg s_*$, the wave variables scale in the same way:

$$F_0(\omega) \equiv F(s_0;\omega): \quad F \sim \omega^{-1/2}\,F_0(\omega) \sim G, \tag{82}$$

Distance	$s \ll s_*$	$s \gg s_*$
$\log F/\log s$	1.00	$1-\nu/2 = +3.19$
$\log G/\log s$	$2\nu-1 = -9.75$	$\nu/2-1 = -3.19$
$\log E_v/\log s$	$2\nu-2 = -10.75$	$\nu-1 = -5.37$
$\log E_h/\log s$	$4\nu-2 = -19.50$	$\nu-1 = -5.37$
$\log D/\log s$	$2\nu = -8.75$	0

TABLE 1. The exponents for the scalings as a function of distance for wave variables F, G and energies E, D for bounded standing waves

and the energy densities like their square:

$$s \gg s_*: \quad E_v, E_h, D \sim \omega^{-1}\{F_0(\omega)\}^2 \equiv E_0. \tag{83}$$

At short distance $s \ll s_*$, there are distinct scalings for the velocity and magnetic field perturbations of bounded waves:

$$s \ll s_*: \quad F \sim F_0(\omega), \quad G \sim \omega F_0(\omega), \tag{84a, b}$$

and hence also for the kinetic and magnetic energy densities and energy flux:

$$E_v, E_h, D \sim \{F_0(\omega)\}^2\{1, \omega^2, \omega\}. \tag{85a–c}$$

Thus the change in the spectrum from small to large distances is given by

$$E_v, E_h D \sim E_0\{\omega, \omega^3, \omega^2\}. \tag{86a–c}$$

This is independent of the exponent (54*a*) which appears in the scaling for distance.

In order to calculate (54*a*), we start with the scale height (54*b*) at the solar corona, where the temperature (Athay 1977) is $T = 1.80 \times 10^6$ K, and the acceleration due to gravity $g_0 = 2.74 \times 10^4$ cm s^{-2}. Using $R = 8.31 \times 10^7$ cm^2 s^{-2} K^{-1} for the gas constant in c.g.s. units, we obtain a scale height $L = 5.46 \times 10^9$ cm at the corona. If we assume that the Alfvén wave starts at the surface of the sun, then the initial distance $r_0 = R_\odot = 6.96 \times 10^{10}$ cm is the solar radius, and since there the magnetic field is radial, $\phi = 0$ or $k = 0$ in (6*a*), we have $r_0 = s_0$ in (18). Thus the polytrophic index in the density law (54*a*) is $n \equiv g_0 r_0/RT = r_0/L = 12.7$ and (54*a*) gives $\nu = 2-n/2 = -4.37$. We can then find the exponents for the scalings as a function of distance for wave variables (68*a*, *b*), (71*a*, *b*) and energies (77*a*, *b*), (79*a*), for bounded standing waves at short distance $s \ll s_*$, and also at large distance $s \gg s_*$; these are written in table 1.

In the case of the solar wind near the Earth, there is a flow with velocity larger than the Alfvén speed. Since we omit this essential feature, we do not claim that the present theory is a model of Alfvén waves in the solar wind: we merely use solar wind data to estimate the parameters in the theory, as an illustration of how it can be applied. Our treatment also assumes that we can replace Parker's spiral (Parker 1959, 1979), which makes an angle with the radial direction which increases with distance, by a logarithmic spiral, for which that angle is constant; this is acceptable if the local wavelength λ is much smaller, $\lambda^2 \ll l^2$, than the length scale l of change of angle of Parker's spiral. In order to estimate the latter we note that the solar magnetic field is radial, $\phi_0 = 0°$, at the corona $s_0 \equiv R_\odot = 6.96 \times 10^{10}$ cm, and makes an angle $\phi_1 = 56°$ with the radial direction at the Earth, $s_1 = 1AU = 215\, R_\odot = 1.50 \times 10^{13}$ cm. Thus the lengthscale for change in the angle of Parker's spiral is

$$\phi_0 = 0: \quad l = \phi\, \mathrm{d}s/\mathrm{d}\phi \sim (\phi_0+\phi_1)(s_1-s_0)/2(\phi_1-\phi_0) = (s_1-s_0)/2, \tag{87}$$

	$\tau = 1$ s	$\tau = 1$ min	$\tau = 1$ h	$\tau = 1$ day
$\phi = 0°$:	-5.50×10^3 (7.16)	-9.17×10 (2.81)	-1.53×10 (1.10)	-6.37×10^{-2} (0.533)
$\phi = 28°$:	-7.06×10^3 (7.58)	-1.18×10^2 (2.97)	-1.96×10 (1.17)	-8.16×10^{-2} (0.564)
$\phi = 56°$:	-1.76×10^4 (9.34)	-2.93×10^2 (3.66)	-4.89×10 (1.44)	-2.04×10^{-1} (0.695)

TABLE 2. The dimensionless frequency and (in brackets) the distance s_* in solar radii beyond which reflection becomes negligible, for four wave periods and three angles ϕ of the magnetic field to the radial direction

approximately half the distance from the sun to the Earth, $l = 7.48 \times 10^{12}$ cm. The condition $\lambda^2 \ll l^2$ is thus met for local Alfvén wavelengths not more than one-sixth the distance from the Earth to the sun, $\lambda < l/3 = 2.49 \times 10^{12}$ cm, and this implies a wave period $\tau < \lambda/A$, where A is the Alfvén speed. The latter varies considerably between the corona and the Earth because: (i) the particle density decays from $N_0 = 5 \times 10^8$ cm^{-3} to $N_1 = 15$ cm^{-3} implying, for a proton mass $m = 1.67 \times 10^{-24}$ g, mass density decaying from $\rho_0 = N_0 m = 8.35 \times 10^{-16}$ g cm^{-3} to $\rho_1 = N_1 m = 2.50 \times 10^{-23}$ g cm^{-3}; (ii) the magnetic field decays from $B = 12\ G$ to $B_1 = 5 \times 10^{-5}\ G$, faster than the square root of mass density $\rho^{1/2}$, and thus the Alfvén speed decays from $A_0 = B_0/(4\pi\rho_0)^{1/2} = 1.17 \times 10^8$ cm s^{-1}, to $A_1 = B_1/(4\pi\rho_1)^{1/2} = 2.82 \times 10^6$ cm s^{-1}, with a geometric mean $A_2 = (A_0 A_1)^{1/2} = 1.82 \times 10^7$ cm s^{-1}. Thus the approximation holds for periods up to $\tau_0 < \lambda/A_0 = 2.13 \times 10^4$ s $= 5.91$ h at the corona, or up to $\tau_1 < \lambda/A_1 = 8.85 \times 10^5$ s $=$ 10.2 days at the Earth, with $\tau_2 < \lambda/A_2 = 1.37 \times 10^5$ s $= 1.59$ days as the geometric mean. Thus we consider four wave periods of interest in the solar wind, namely 1 s, 1 min, 1 h and 1 day, and three angles of the magnetic field to the radial direction: $\phi = 0°, 28°, 56°$, and indicate the nine corresponding values of (i) the dimensionless frequency (56*b*) and (ii) the distance $s_* = R_\odot \Omega^{-0.229}$ in solar radii (69), beyond which wave reflection becomes negligible, and the spectrum does not change any more. These are listed in table 2.

7. Discussion

We plot the velocity (81*a*) and magnetic field (81*b*) perturbations, in dimensionless form, normalized by their initial value at $s = s_0$:

$$V_\pm(R) = RH_0^{(1,2)}(R^{-4.4}\Omega)/H_0^{(1,2)}(\Omega), \tag{88a}$$

$$H_\pm(R) = R^{-5.4}H_1^{(1,2)}(R^{-4.4}\Omega)/H_1^{(1,2)}(\Omega), \tag{88b}$$

for an outward $H^{(2)}$ or inward $H^{(1)}$ propagating wave, which have the same amplitudes and opposite phases:

$$V \equiv V_+: \quad |V| = |V_\pm|, \quad \arg(V_+) = \arg(V) = -\arg(V_-), \tag{89a}$$

$$H \equiv H_+: \quad |H| = |H_\pm|, \quad \arg(H_+) = \arg(H) = -\arg(H_-). \tag{89b}$$

The general solution (62*b*) is a linear combination of inward and outward propagating waves, with amplitudes $C_\pm$ specified by boundary conditions. The distance is normalized by dividing by the initial distance (or solar radius):

$$1 \leqslant R \equiv s/s_0 = r/r_0 \leqslant 215. \tag{90}$$

The only parameter is the dimensionless frequency (56*b*):

$$\Omega \equiv \Omega_0 |\sec^2\phi|, \quad \Omega_0 \equiv s_0\omega/\nu a, \tag{91a, b}$$

to which we give six values:

$$-\Omega_0 = 3 \times 10^{-2}, 3 \times 10^{-1}, 3, 3 \times 10, 3 \times 10^2, 3 \times 10^3, \tag{92a}$$

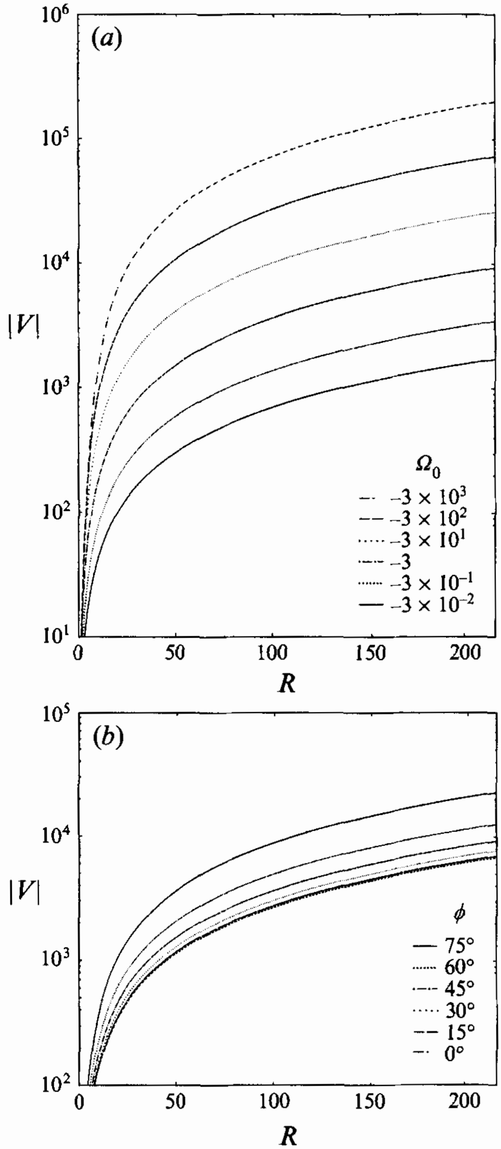


FIGURE 2. Modulus of velocity perturbation (88*a*), (89*a*), normalized by the value at the solar surface, versus radial distance normalized by solar radius, for Alfvén waves, propagating along a logarithmic spiral: (*a*) making an angle $\phi = 45°$ with the radial direction, for five values of dimensionless frequency; (*b*) for fixed dimensionless frequency $\Omega_0 = -3$ and five values of spiral angle.

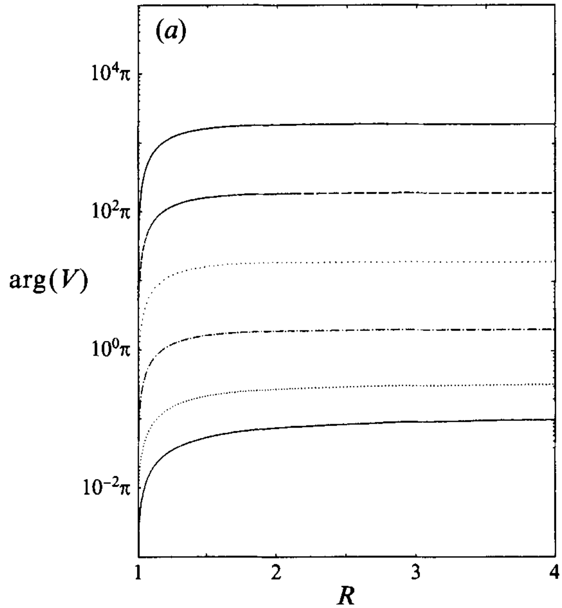
corresponding respectively to periods $\tau = 2\pi/\omega = -2\pi R_{\odot}/v a \Omega_0 = 1.83$ s, 18.3 s, 3.06 min, 30.6 min, 5.09 h, 2.12 days, in the case of the solar wind; we also give six values to the angle of the spiral with the radial direction:

$$\phi = 0, 15°, 30°, 45°, 60°, 75°. \qquad (92b)$$

We take as baseline case

$$\Omega_0 = -3, \quad \phi = 45°, \quad \Omega = -6, \qquad (93a\text{–}c)$$

corresponding to a period of 3.06 min and an equilateral spiral and vary each of the two parameters (92*a, b*) in turn. Thus we separate the effects of angle of the spiral



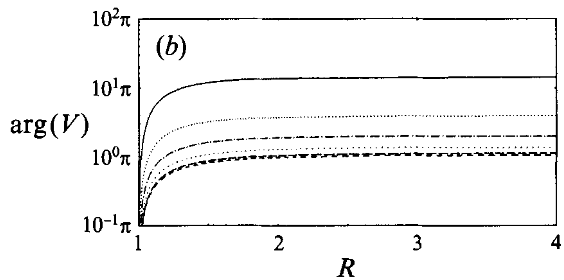


FIGURE 3. As figure 2 but for the phase difference of the velocity perturbation measured from the solar surface.

magnetic field ϕ and the stratification by presenting pairs of plots, one in which ϕ is fixed and Ω_0 varies, and vice versa in the other. The separation of the wave fields, both for the velocity and magnetic field perturbations, into amplitude and phase also helps to visualize the effects of spiral angle and stratification.

Figure 2 shows the modulus of the velocity perturbation (89*a*), normalized by the initial value (88*a*), at the solar surface $r = R_\odot$ versus radial distance to the Earth, also normalized (see (90)): it applies both to outward $H_0^{(2)}$ and inward $H_0^{(1)}$ propagating waves. It is seen that the wave amplitude increases outwards, as density decreases, and: (i) for a fixed spiral angle (figure 2*a*), the amplitude is larger for higher frequency; (ii) for fixed dimensionless frequency (figure 2*b*), the amplitude is larger for greater spiral angle, i.e. longer distance of propagation. The first effect is more pronounced, in the sense that it causes a greater spread of amplitudes. The phase difference of the velocity perturbation (88*a*) between an arbitrary radius and the solar 'surface' has opposite signs (89*a*) for inward and outward propagating waves. If the exponent ν in (54*a*) were positive, $\nu > 0$, the phase change would increase with distance, as can be confirmed in the case of a radial magnetic field (Campos 1994); since the exponent is negative in the present case, (88*a*), phase changes decay with distance, i.e. the phase tends to a constant value in figure 3 in all cases. Since most of the phase change occurs within a few solar radii, we have plotted the phases (figures 3 and 5) over 4 solar radii, instead

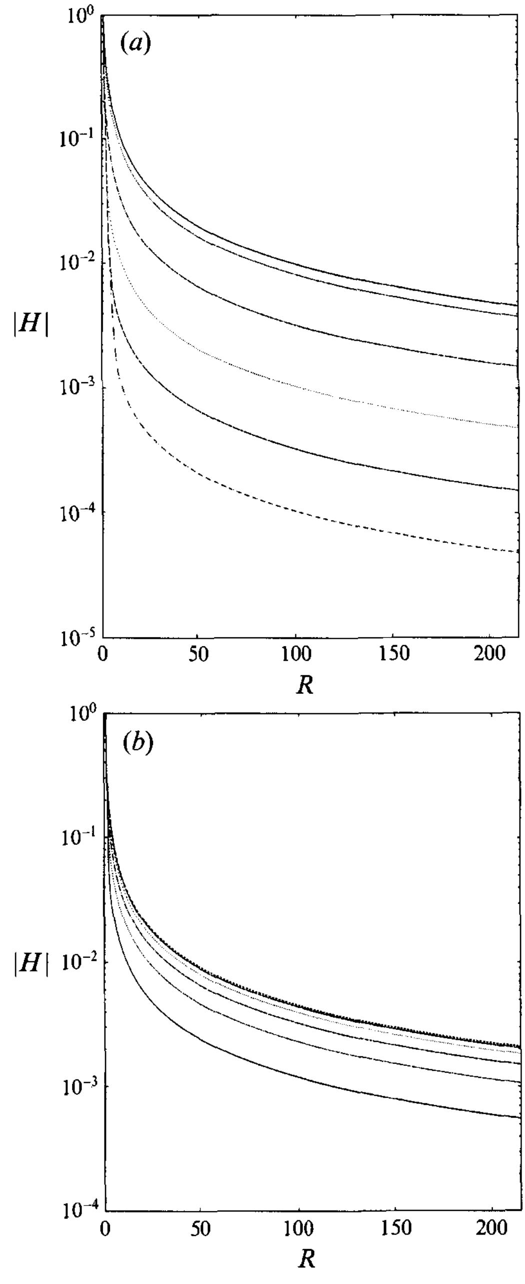


FIGURE 4. As figure 2 but for modulus of the magnetic field perturbation (88 *b*), (89 *b*) normalized to value at solar surface.

of over the sun to Earth distance of 215 solar radii for amplitude (figures 2 and 4); the dependent variable (amplitude or phase) is in a logarithmic scale in all cases. The phase of the velocity perturbation increases with dimensionless frequency (figure 3 *a*) and spiral angle (figure 3 *b*), the former effect being more pronounced.

The amplitude of the magnetic field perturbation normalized by the value at the solar surface (88 *b*), is the same, (89 *b*), for inward and outward propagating waves, and decays with radial distance (figure 4); this contrast with the velocity perturbation,

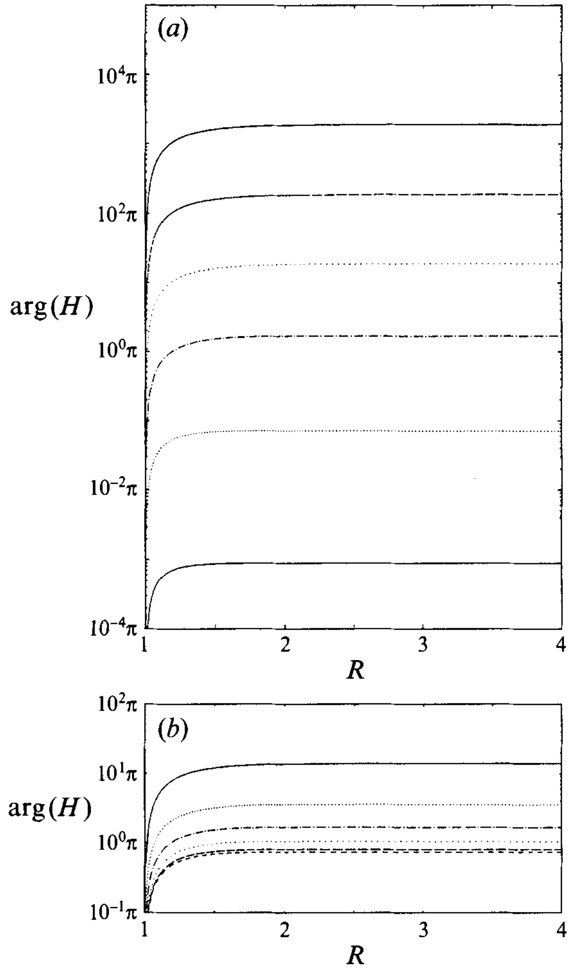


FIGURE 5. As figure 2 for phase of magnetic field perturbation measured from solar surface.

which increases in amplitude with radial distance (figure 2), is related to the equipartition of kinetic and magnetic energies (79*a*), which holds at large distances. From $\rho V^2 \sim \mu H^2/8\pi$ it follows that $H \sim V\rho^{1/2}$, and since the square root of mass density of the medium decays faster (exponent $\nu - 2 = -6.35$ in (76)) than the velocity perturbation of the Alfvén wave increases (exponent $1-\nu/2 = +3.19$ in table 1), the magnetic field perturbation of the Alfvén wave decays with distance. The decay is more rapid for increasing frequency (figure 4*a*) and spiral angle (figure 4*b*), with the effect of the former being more pronounced. The frequency (figure 5*a*) also has a more pronounced effect than spiral angle (figure 5*b*), for the phase change of the magnetic field perturbation between the solar surface and arbitrary radius. The evolution of phases is broadly similar for the velocity (figure 3) and magnetic field (figure 5) perturbations of the Alfvén wave, whereas it is opposite for the amplitudes (figures 2 and 4).

We conclude the illustrations with plots (figure 6) of spiral coordinate curves for the five cases considered for the amplitudes and phases of the velocity and magnetic field perturbations of Alfvén waves. We start (figure 6*a*) with the case $\phi = 45°$, where both sets of logarithmic spirals (figure 1) make the same angle (in modulus, and with

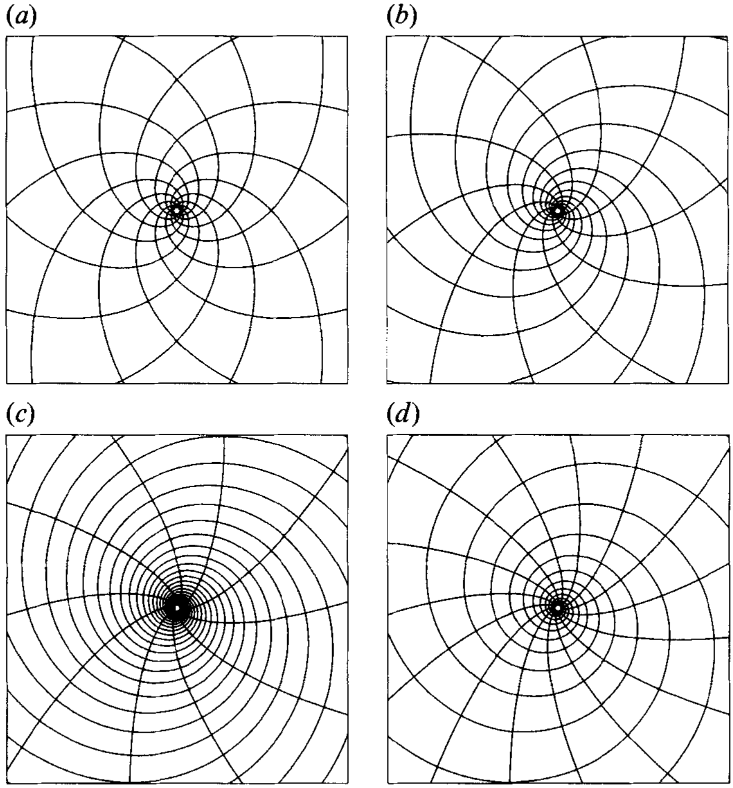


FIGURE 6. Orthogonal logarithmic spiral coordinates where one set of spirals makes an angle with the radial direction equal to: (*a*) $\phi = 45°$, (*b*) $\phi = 30°$, and (*c*) $\phi = 15°$, and ten coordinate curves are used for each coordinate. For (*d*) we take $\phi = 15°$ as in (*c*), but with five curves of one coordinate and 15 of the other.

opposite signs) with the radial direction. The case (figure 6*b*) of one set of spirals making an angle $\phi = 30°$ with the radial direction is the mirror image of $\phi = 60°$; similarly the cases $\phi = 15°$ (figure 6*c*) and $\phi = 75°$ are also mirror images. In the first three plots (figure 6*a–c*) ten coordinate curves were drawn for each coordinate, giving a symmetrical appearance for $\phi = 45°$, and becoming more unsymmetrical for $\phi = 30°$ and $\phi = 15°$, as one set of coordinate curves curls more and the other less; this gives the impression (in figure 6*c*) of coordinate curves of one set closer together than those of the other set, although in fact the number is the same (ten for each). In figure 6(*d*) the two sets of coordinate curves appear almost equally spaced for $\phi = 15°$, because we have plotted 15 curves of the set $\alpha(\phi = 15°)$ and only 5 of the orthogonal set $\beta(\phi = 75°)$.

The present paper contains a solution of the Alfvén wave equation in a magnetic field which varies both in strength and direction, and in an inhomogeneous medium as well, as required by (magneto)hydrostatic equilibrium; other solutions in inhomogeneous media under non-uniform magnetic fields appear in the literature (Whang 1973; Velli 1993; Oliver *et al.* 1993; Lou 1994). The Alfvén wave equation was first obtained (Alfvén 1942, 1948) for a homogeneous medium under a uniform magnetic field, followed by the extension to a plane parallel atmosphere under a uniform magnetic field (Ferraro & Plumpton 1958, 1963; Hollweg 1972, 1978; Leroy 1980; Campos 1983*a*, *b*, 1987) and to magnetic flux tubes (Roberts 1981; Spruit 1982). Dissipative effects have been considered in homogeneous media (Cowling 1960;

Moffatt 1978) and atmospheres, the latter using the phase mixing approximation (Heyvaerts & Priest 1983; Nocera, Leroy & Priest 1984) or exact solutions (Campos 1983*c*, 1988, 1989, 1993*a, b*). Mean flow effects have been included using the ray approximation (Belcher 1972; Whang 1973; Hollweg 1973; McKenzie, Ip & Axford 1983). Alfvén waves with the Hall effect illustrate well the difference between homogeneous media (Lightill 1959), and inhomogeneous media treated in the ray approximation (McKenzie 1979), or by means of full wave solutions, the latter with uniform (Campos 1992) and non-uniform (Campos & Isaeva 1993) magnetic fields. One of the motivations for these studies has been the observation of Alfvén waves in the solar wind (Belcher 1971; Denskat & Burlaga 1977).

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Appendix. On conformal coordinate systems

We define a conformal coordinate system (α, β) as one specified by a complex analytic function:

$$w = \beta + \mathrm{i}\alpha, \quad w = f(z), \tag{A 1a, b}$$

in terms of Cartesian (x, y) or polar (r, θ) coordinates:

$$z = x + \mathrm{i}y = r\,\mathrm{e}^{\mathrm{i}\theta}. \tag{A 2a, b}$$

The arclength in Cartesian or polar coordinates:

$$(\mathrm{d}l)^2 \equiv |\mathrm{d}z|^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2, \tag{A 3a, b}$$

becomes

$$|\mathrm{d}z|^2 = |f'(z)|^{-2}\,|\mathrm{d}w|^2 \tag{A 4}$$

in conformal coordinates

$$|\mathrm{d}l|^2 = s^2\{(\mathrm{d}\alpha)^2 + (\mathrm{d}\beta)^2\}, \quad s \equiv 1/|f'(z)|, \tag{A 5a, b}$$

where the scale factor is the inverse of the modulus of the derivative of the function. Note that, since the function $f(z)$ is holomorphic: (i) the derivative $f'(z)$ exists; (ii) it is independent of direction. This proves that for a conformal coordinate system the scale factor is the same for both coordinate curves. This is a consequence of the fact that a conformal transformation preserves angles, i.e. all lengths are multiplied by the same factor in every direction.

Spiral coordinates are the particular case of conformal coordinates specified by

$$w = (1 - \mathrm{i}k)\log z \equiv f(z). \tag{A 6}$$

Using (A 1) and (A 2*b*):

$$\beta + \mathrm{i}\alpha = (1 - \mathrm{i}k)(\log r + \mathrm{i}\theta), \tag{A 7}$$

we obtain the direct spiral coordinate transformation (4*a, b*). The inverse of (A 6):

$$z = \exp\{w/(1 - \mathrm{i}k)\}, \tag{A 8}$$

leads through (A 1) and (A 2*b*):

$$r\,\mathrm{e}^{\mathrm{i}\theta} = \exp\{(\beta + \mathrm{i}\alpha)(1 + \mathrm{i}k)/(1 + k^2)\}, \tag{A 9}$$

to the inverse spiral coordinate transformation (14). The derivative of (A 6):

$$f'(z) = (1 - \mathrm{i}k)/z, \tag{A 10}$$

specifies the scale factor

$$s = |z|/|1-ik| = r/(1+k^2)^{1/2}, \tag{A 11}$$

as in (18); substitution of (A 11) into (A 5*a*) yields the arc element in spiral coordinates (15).

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